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Allocation in an Uncertain World: Maximising the Use of Data with UBA on Global Producer III

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1 INTRODUCTION

This paper describes the application of non-linear Uncertainty Based Allocation (UBA) to allocate oil and gas between the Dumbarton and Lochranza fields produced across Maersk's Global Producer III (GPIII) FPSO (Floating Production Storage and Offloading).

All Lochranza wells' production is measured continuously using individual subsea multiphase flow meters (MPFMs). However, Dumbarton wells' production is only available from estimates derived from subsea multiphase well tests. Hence the uncertainty in Lochranza's production is significantly better than Dumbarton's. However, rising water cuts and the increasing presence of gas lift started to produce relatively high uncertainties in calculated oil and produced gas respectively for all MPFM measurements.

In this challenging measurement environment, the application of UBA, allows the maximal use of all available, pertinent data, including field GORs, to allocate oil and gas products simultaneously in a robust fashion, whilst optimising allocation uncertainty.

The approach is described using simple theoretical examples and illustrated with real data supplied by Maersk. The data from the real system covers a five and half year period (Jan 2007 to July 2012). In the first three years, only Dumbarton wells were flowing (Jan 2007 to Jan 2010) before Lochranza commenced production.

Section 2 describes the GPIII subsea configuration and topsides process, the Dumbarton and Lochranza fluids and the previous, historical allocation system. It also describes some of the issues encountered with the old system which prompted the investigation to examine alternatives. Section 3 describes the alternative allocation approaches analysed and examines their performance whilst Section 4 provides a direct comparison of all the approaches considered in terms of impact on allocated quantities and associated uncertainties. Finally Section 5 provides some conclusions on the use of UBA on GPIII.

2 DESCRIPTION OF SYSTEM

2.1 Process

A schematic of the sub-sea well configuration is presented in Figure 1:

Figure 1 – Dumbarton, Lochranza Sub-Sea Configuration

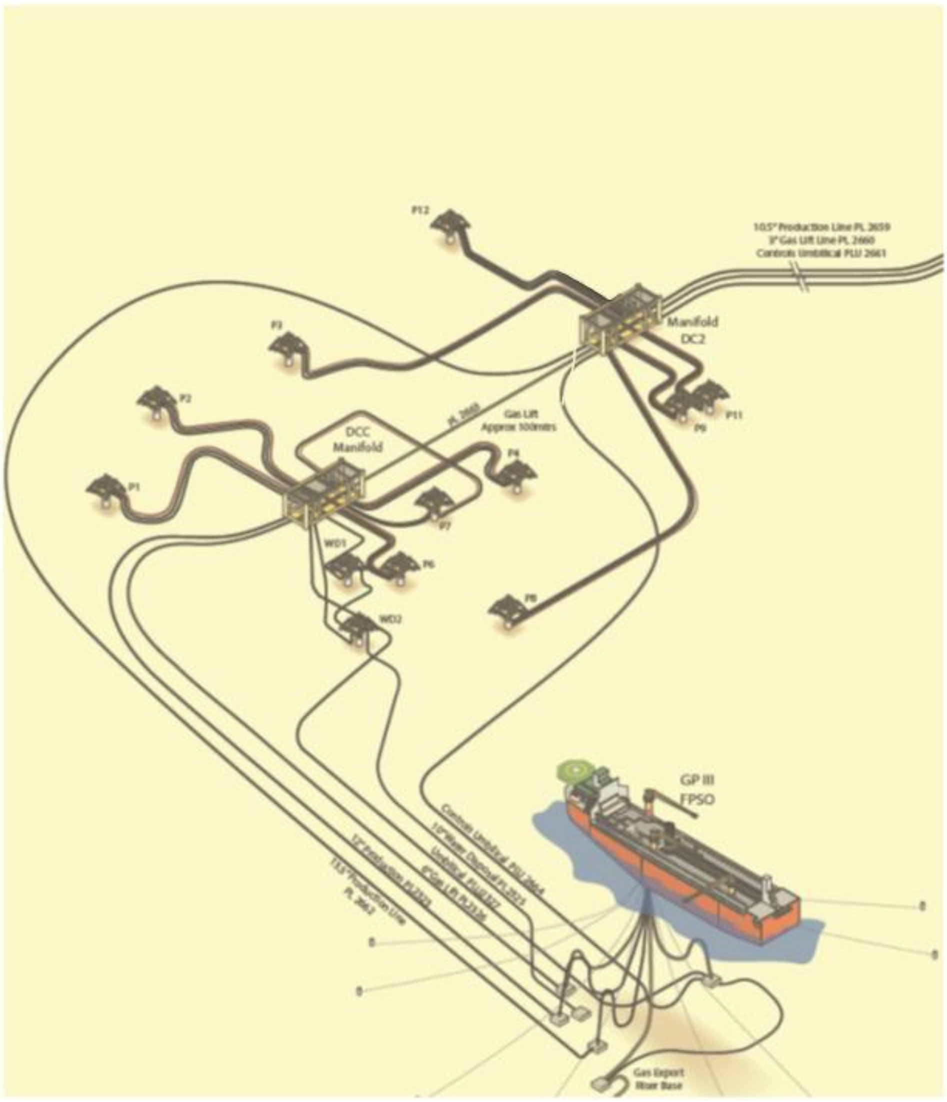
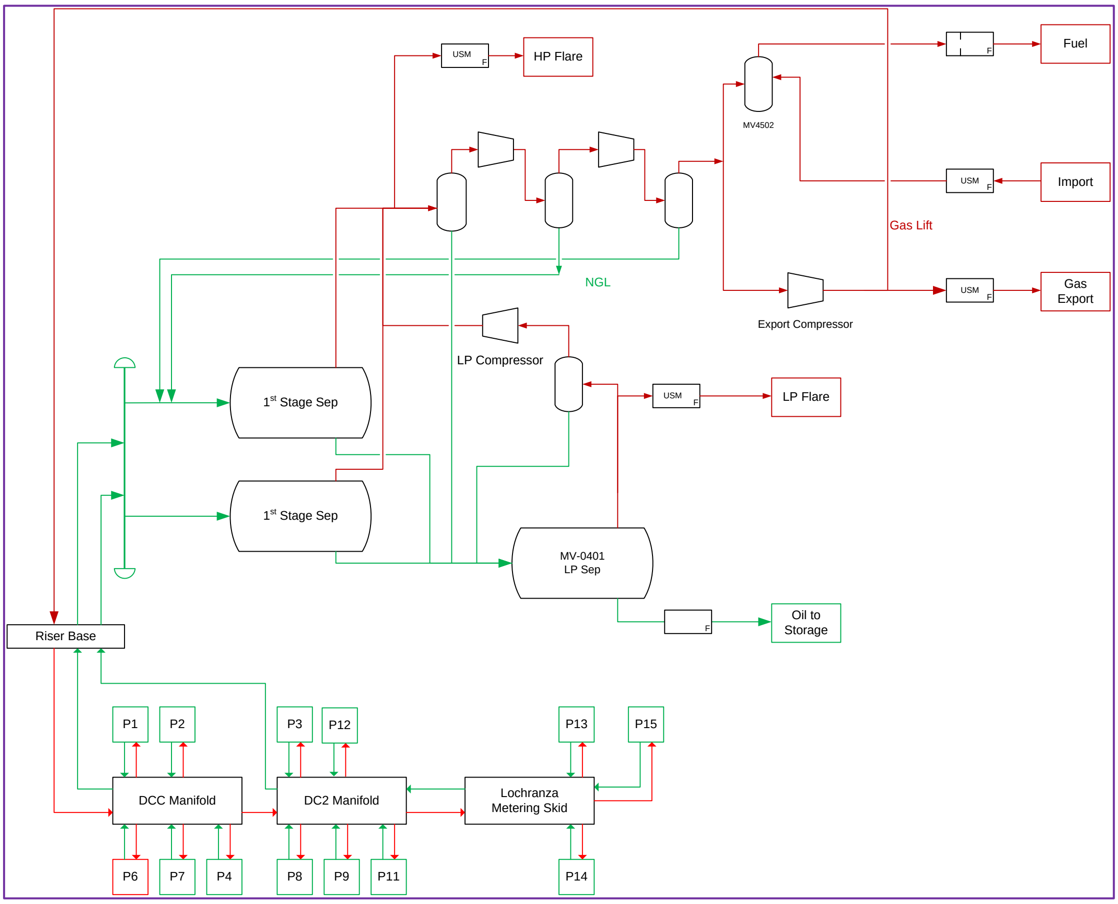


Figure 2 shows the subsea and GPIII topsides process and associated topsides metering:

Figure 2 – GPIII: Simplified Schematic of Subsea and Topsides Process



The GPIII FPSO handles production from both the Dumbarton and Lochranza fields. Production is metered with a combination of export product meters and subsea multiphase flow meters (MPFM). While each of the Lochranza wells has a dedicated MPFM, there is a single MPFM for each of the Dumbarton drill centres (DCC & DC2) which are used to test the performance of the Dumbarton wells. Lift gas to each Lochranza well is also individually metered.

As shown in Figure 2, the Dumbarton and Lochranza fluids are commingled upstream of the 1st stage separators.

Oil separation on the GPIII is achieved using two-stage separation with inlet and 2nd stage heating. Gas from separation is sent to the Low Pressure (LP) and High Pressure (HP) compression trains with produced water passed to the produced water handling package.

Currently, the plant is recycling large quantities of NGL from the compression trains to the separators. This has presented difficulties with the modelling of the process in simulation packages and is discussed further in see Section 2.3.

The oil is stabilised and offloaded by tanker and shipped to market. The export gas is transported via the MacCulloch FPSO tie-in, onto Piper B and into the Frigg system at St Fergus.

All oil and gas product streams (including fuel and flare) are measured.

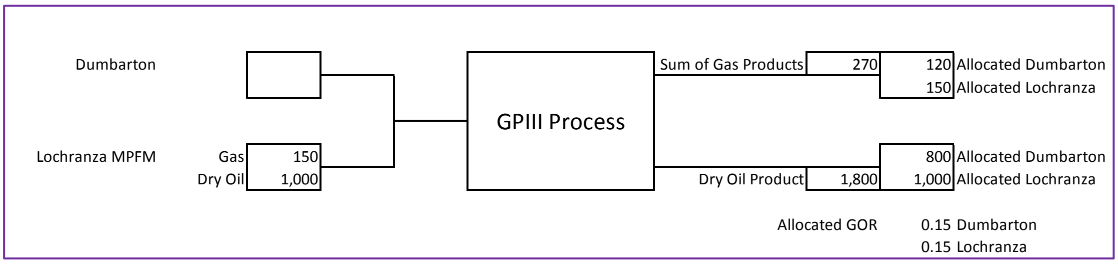
2.2 Historical Allocation Scheme

Historically, once Lochranza commenced flowing the Dumbarton field was allocated By-Difference. For example the Dumbarton's allocated oil was determined by subtracting the totalised Lochranza MPFM dry oil flow, after allowing for shrinkage, from the commingled oil export meter.

Similarly, the total produced gas was calculated by summing fuel, flare, export (and netting import) measured flows and subtracting the Lochranza MPFM gas flow, after allowing for lift and process effects, to obtain Dumbarton allocated gas.

The calculations are illustrated using a simple example in Figure 3:

Figure 3 – Simplified Example: Historical Dumbarton By-Difference Allocation



The figures can nominally be considered to be in tonnes but they are presented merely for illustrative purposes and hence no units are shown.

Lochranza is allocated an oil product quantity equal to its MPFM measured dry oil (1,000 units) – processing effects or shrinkage have been ignored to render the calculations simpler. Dumbarton is allocated the difference between the measured export oil and the allocated Lochranza oil (1,800 – 1,000 = 800). Gas product is allocated in a similar fashion.

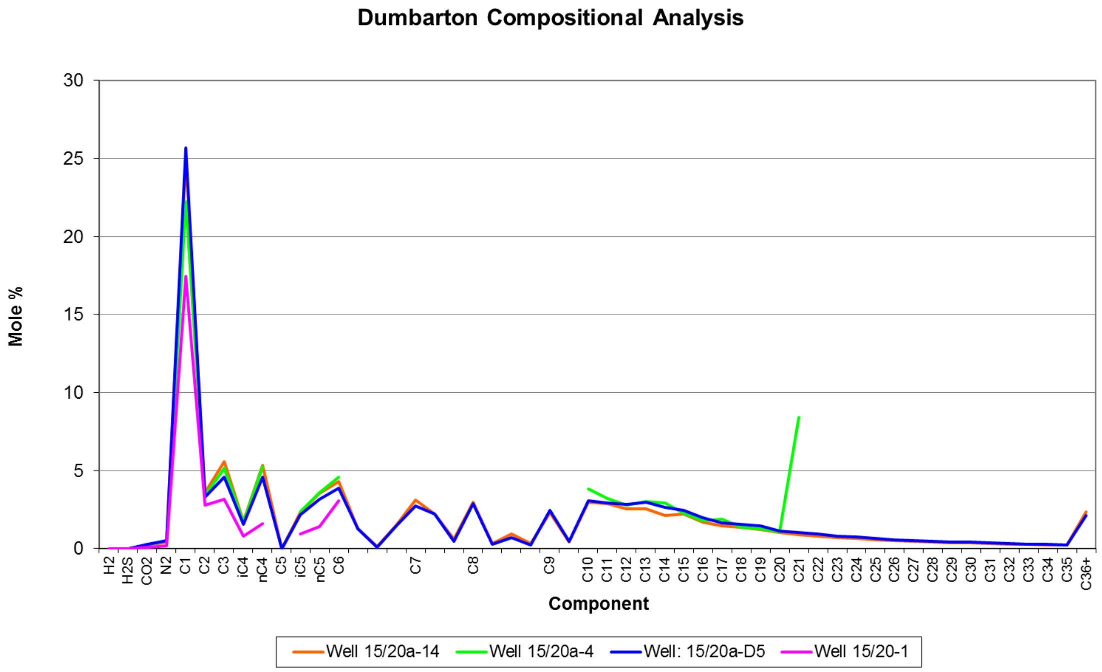
This simple example is utilised later to illustrate alternative allocation methodologies.

2.3 Fluid Compositions

An important feature of both the Dumbarton and Lochranza reservoirs is that their pressure is maintained above the bubble point. This means that the hydrocarbons in the reservoir rock will be in a single phase and hence when produced up the well bore the composition of each field's hydrocarbon fluids entering the GPIII process should be essentially constant.

This is illustrated in Figure 4 below which plots and compares the compositions of samples taken of the various Dumbarton wells (obtained at various times).

Figure 4 – GPIII: Dumbarton Well Sample Analyses

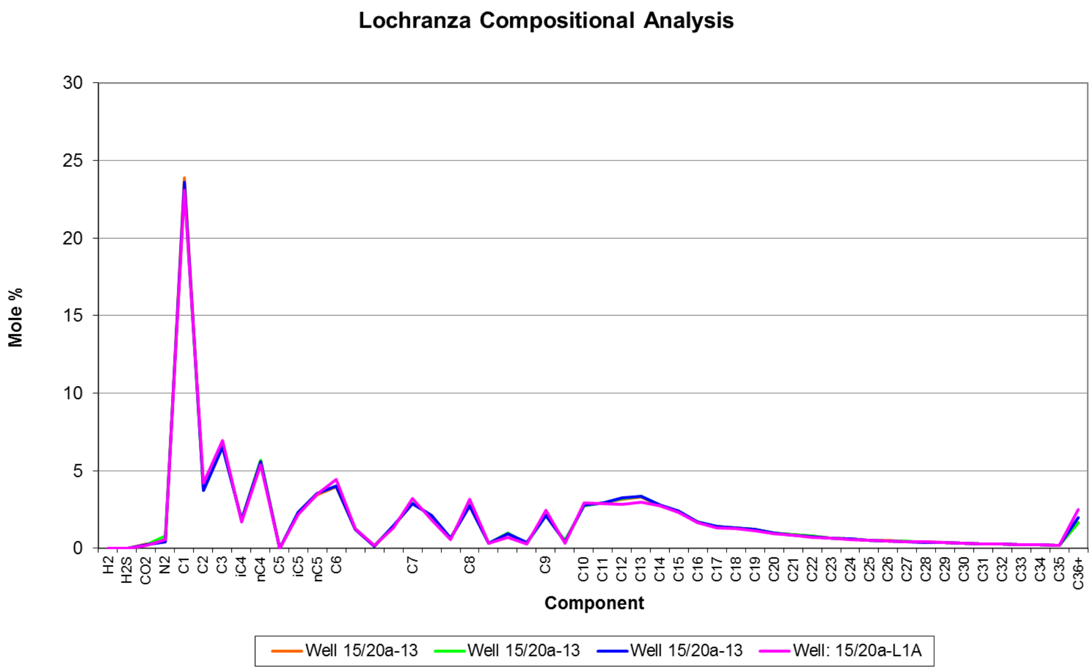


Samples obtained:

- Well 15/20a-14 (red) Jan 2004
- Well 15/20a-4 (green) Aug 1987
- Well 15/20a-D5 (blue) Sep 2006
- Well 15/20-1 (purple) Mar 1975.

And similarly for the Lochranza fluids:

Figure 5 – GPIII: Lochranza Well Sample Analyses



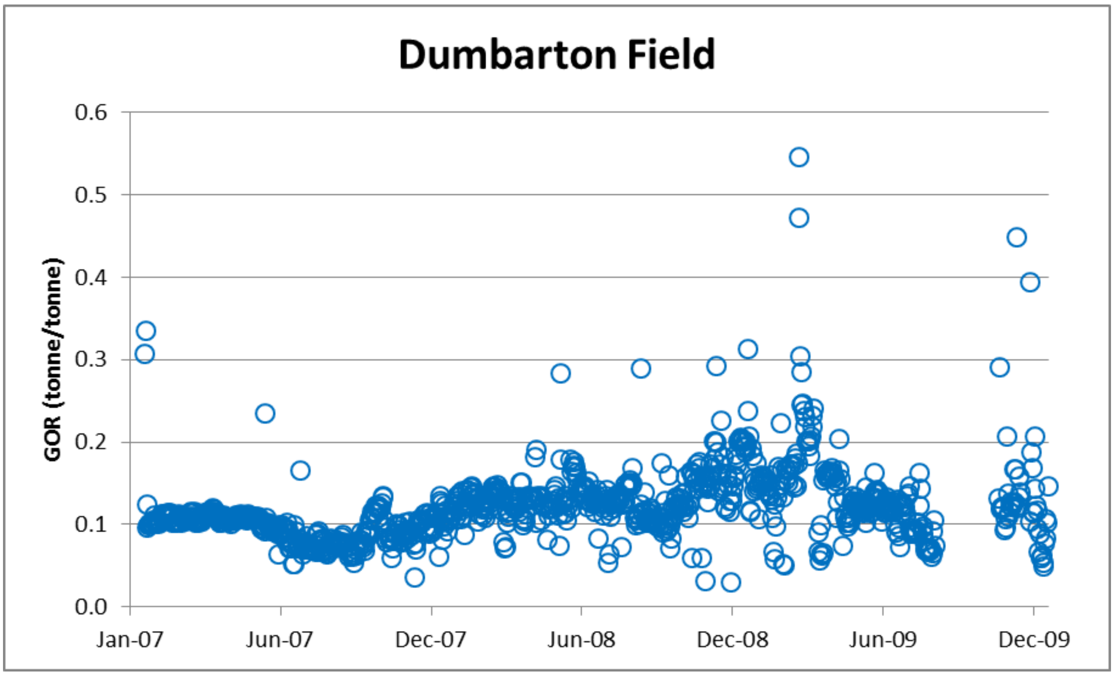
Samples obtained:

- Well 15/20a-13 (red) May 2003
- Well 15/20a-13 (green) May 2003
- Well 15/20a-13 (blue) May 2003
- Well 15/20A-L1A (purple) Sep 2010.

This also means that the Gas Oil Ratio (GOR) of the fields' allocated fluids (ratio of produced gas to produced oil) should remain relatively stable, though some variation is to be expected due to commingling effects and variations in operating conditions in the topsides process. In addition, because of their stability, the allocated GORs should provide a metric with which to monitor the veracity of the allocation system.

A plot of Dumbarton's allocated GOR for the three year period from January 2007 to the beginning of January 2010, when only Dumbarton was produced, is presented in Figure 6:

Figure 6 – GPIII: Dumbarton Allocated GOR
Dumbarton Production Only



The GOR presented is mass based and is calculated as the sum of total measured produced gas mass (fuel, flare, injection and export) divided by the measured export oil mass. Though there are some outliers, the data indicates a reasonably stable GOR. The average, standard deviation and range of values associated with the data in the above plot are presented in Table 1.

Table 1 – Allocated Dumbarton GOR Statistics (Jan 2007 to Jan 2010)

	All Data	Outliers Removed
Average	0.121	0.118
Maximum value	0.545	0.246
Minimum value	0.030	0.030
Standard Deviation (abs)	0.044	0.033
Uncertainty (rel %)	±72%	±57%

Based on all the data, Dumbarton's GOR is estimated to be 0.121 tonne/tonne with a standard deviation of 0.044 tonne/tonne which is equivalent to an estimated uncertainty of ±72% (twice the standard deviation expressed relative to the average). This uncertainty or variability in the GOR appears to be due to a number of factors:

- Measurement uncertainty in the oil and gas product meters
- Variation in the wellstream composition (if any)
- Variation in the GPIII process operating conditions (temperatures and pressures)
- Process instabilities due to high NGL recycle
- Other unknown causes.

These factors are considered in turn below.

Measurement Uncertainty

The uncertainties of the gas and oil product measurements are presented in Table 2:

Table 2 – Oil and Gas Product Meter Uncertainties

Meter	Relative Uncertainty (±%)
Oil Export	0.5%
Gas Export	1%
Fuel Gas	2%
HP Flare	5%
LP Flare	5%
Gas Injection	2%
Gas Import	1%

These figures are typical nominal values appropriate for the type of meter installed. Based on these figures the uncertainty in the Dumbarton GOR due to meter uncertainty alone can be calculated and is predicted to be between ±1% and ±5%, depending on the relative flow rates of oil and gas. (These uncertainties in the GOR have been calculated using the approached described in the GUM [1], termed Taylor Series Method (TSM), which is used to model the propagation of uncertainties. The use of the term "analytical" with reference to uncertainty calculations denotes this TSM method. This is to distinguish that approach from the Monte Carlo Method (MCM), which is described in a Supplement to the GUM [2]).

It is possible that the quoted meter uncertainties are optimistic but it appears unlikely that the variability in GOR is due to measurement uncertainty alone.

Compositional and Processing Uncertainty

Based on a single stage stock tank flash of the Dumbarton field composition the GOR is predicted to be 0.113 tonne/tonne, which is in good agreement with the average allocated GOR.

However, when simulated in the full multi-stage topsides process, the GOR is predicted to be around 0.075 tonne/tonne, significantly below the average observed figure. The reason for this is considered further in the section on process instability below.

In order to estimate the impact of wellstream compositional measurement uncertainty and variability in process operating conditions, the steady state topsides process model was used as part of a Monte Carlo simulation. In this simulation, the feed compositions and process operating conditions were randomly varied within known process limits, the process model solved and the variation in the resultant modelled GOR calculated. It was found that this produced a variation consistent with an uncertainty of approximately ±10%.

Process Instabilities

The observed variation in GOR is too great to be accounted for by compositional and flow measurement uncertainties or variations in steady state process operating conditions alone.

As mentioned above the when simulated in the full multi-stage topsides process the GOR is predicted to be significantly below the average observed figure.

A problem with the full process simulation model is that in order to reach steady state thermodynamic equilibrium in all the vessels, the NGL recycles have to reach unfeasibly high flow rates. This means that steady state equilibrium is not being established in the actual process. Failure to establish equilibrium would most likely result in carryover of liquid in the gas through the process and hence would probably tend to result in a higher GOR than predicted by the steady state model. This possibly also accounts for a good deal of the variation observed in the measured GOR.

Dumbarton GOR and Uncertainty

It appears that a significant proportion of the observed uncertainty is due to process instability and other unknown factors (which can effectively be lumped together).

However, despite these issues the data appears to confirm the hypothesis that the GOR is essentially stable in accordance with an ostensibly constant Dumbarton wellstream composition resulting from the fact that the reservoir is above its bubble point and the hydrocarbons being single phase therein.

The observed variation in the GOR is due to a number of factors discussed above, but based on the historical data, the GOR can be stated to be a nominal value albeit with an expected variability or uncertainty determined from the standard deviation of the observed data. Analysis of that data however, does reveal a number of points that appear to be outliers and these are apparent in Figure 6.

Outliers can be identified statistically using the Grubbs' Test[1] [4]. This was applied to the Dumbarton GORs presented in Figure 6; 14 out of 979 data points were identified as outliers and removed from the data used to calculate the average GOR and uncertainty. The second column of Table 1, presents the statistics for the data set but with outliers removed resulting in an average GOR of 0.118 tonne/tonne with an estimated uncertainty of ±57%.

2.4 Lochranza and Dumbarton Allocation Data

When Lochranza wells (P13 and P14) came on stream in January 2011 Dumbarton was then allocated by-difference. As might be anticipated there was an increase in the uncertainty of Dumbarton's allocated gas and oil. The relative uncertainty will increase as Dumbarton production reduces relative to Lochranza. This is typical of a by-difference allocation scheme.

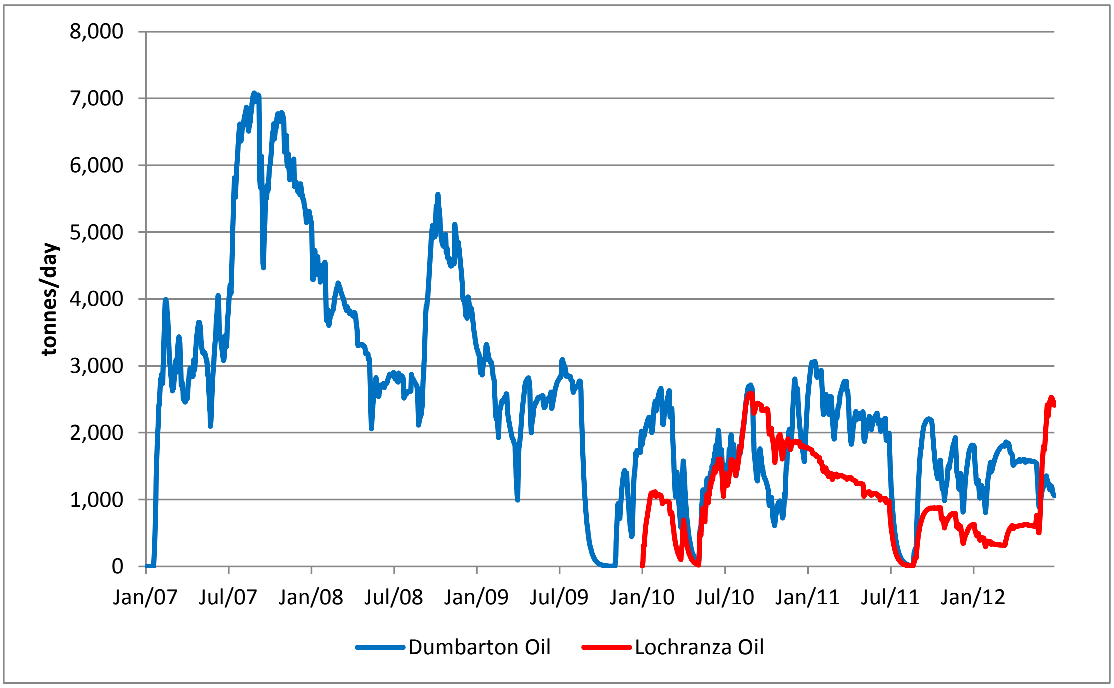
Oil Allocation

The oil flows[2] for Dumbarton and Locranza for the full 5½ year period analysed are presented in Figure 7:

1 The Grubbs' test is a statistical test used to detect outliers in a univariate data set assumed to come from a normally distributed population.

2 Actual data has been filtered to smooth the data points for reasons of clarity.

Figure 7 – Lochranza and Dumbarton Allocated Oil Mass



As can be observed, Dumbarton's oil production is generally declining, though when Lochranza starts up it is the dominant flow so allocation by difference appears a reasonable approach. However, as more Lochranza wells came on stream and its production increased it might be anticipated that Dumbarton's allocation By - Difference would become problematic in terms of allocation uncertainty. This can be observed at the right hand edge of the chart when a third Lochranza well (P15) was started up and Dumbarton became the minor flow.

The problems were exacerbated by the rising water cut of the Lochranza wells which had increased to over 80% for the first two wells (P13 and P14) by the end of the study period. Increased water cuts in an MPFM result in increased relative uncertainty in the dry oil measurement.

The uncertainty in both fields' allocated oil mass can be calculated using the relative field flow rates (Figure 7), uncertainties in the oil export meter (from Table 2) and Lochranza multiphase flow meter measurements. The MPFM uncertainties are reproduced in Table 3:

Table 3 – Lochranza Well MPFM and Lift Gas Meter Uncertainties

Meter	Uncertainty type	Uncertainty (±%)
MPFM Gas Flow	relative	3%
MPFM Liquid Flow	relative	3%
MPFM Water Liquid Ratio (WLR)	absolute	3%
Lift Gas	relative	3%

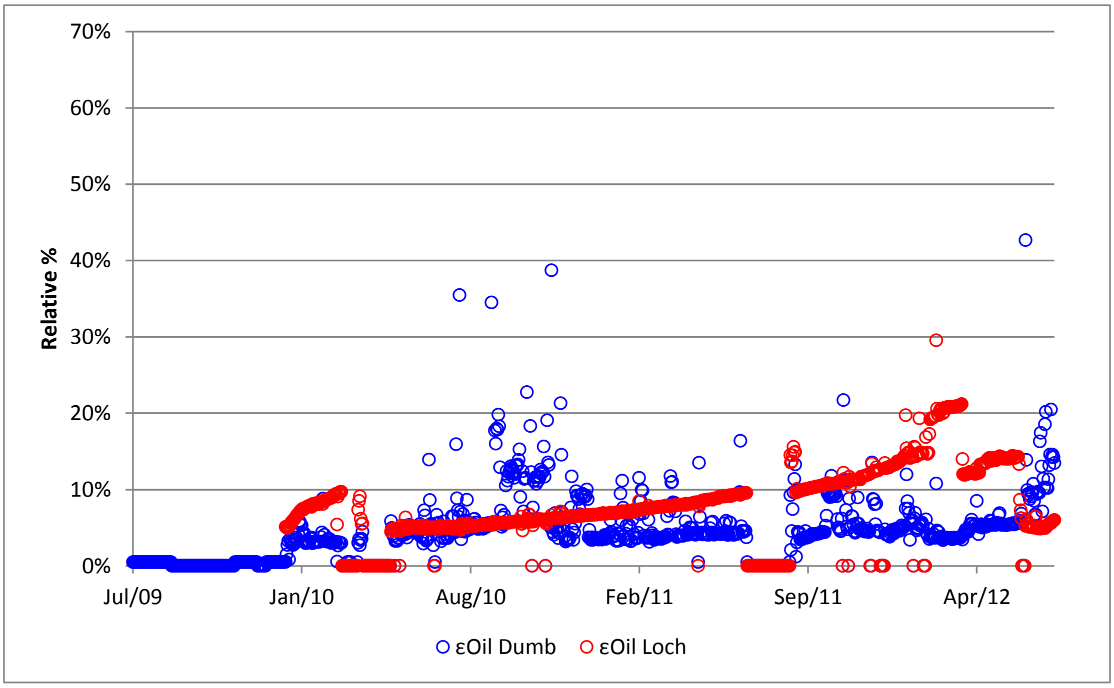
The MPFM uncertainties are based on a paper delivered at the 2010 North Sea Flow Measurement Workshop [3] and are in accordance with a GVF below 90% and operating pressure above 20 barg. For an MPFM the dry oil flow uncertainty is function of the measured liquid and WLR and their associated uncertainties. The relative uncertainty in the oil flow is given by:

$$\varepsilon_{M_{Oil}} = \frac{\sqrt{(\varepsilon_{Liq} * (1 - WLR))^2 + (e_{WLR})^2}}{(1 - WLR)} \quad (1)$$

Inspection of the above equation reveals that the relative uncertainty in the oil becomes very large as the WLR approaches 1.

Using the analytical TSM method, the relative uncertainty in the allocated oil for both fields is presented in Figure 8:

Figure 8 – Lochranza and Dumbarton Allocated Oil Mass Relative Uncertainty



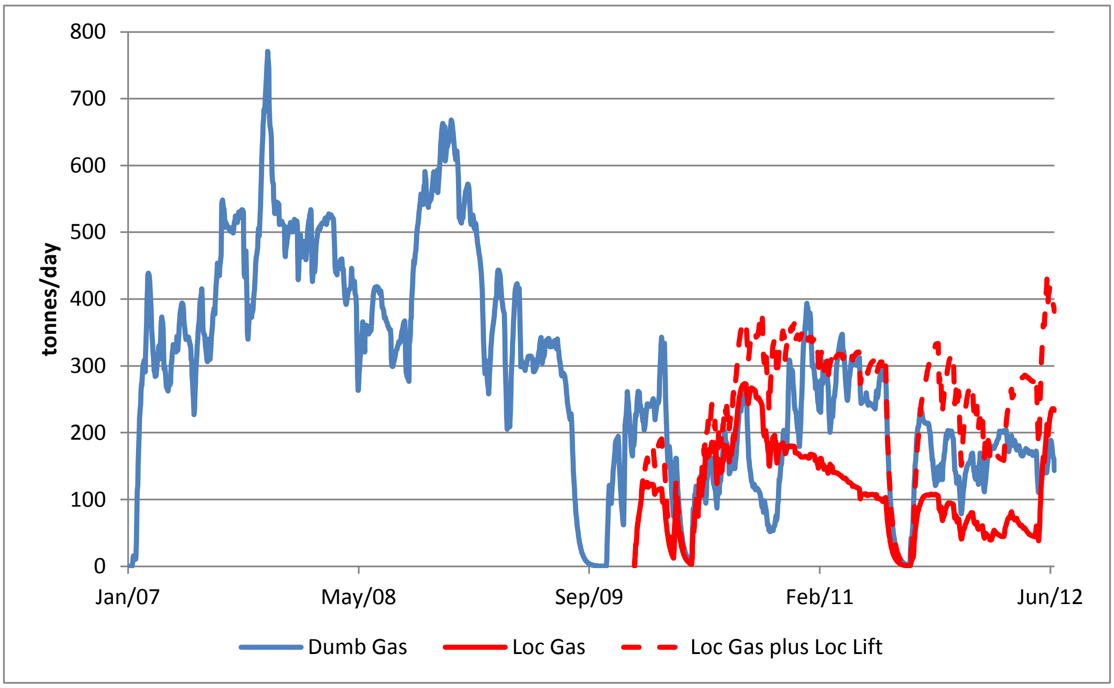
The period presented includes 6 months of Dumbarton only (Jul 2009 to Jan 2010) when its oil allocation uncertainty was that of the oil export meter (±0.5%). Once Lochranza starts up Dumbarton's allocated oil uncertainty experiences a step increase to around ±5%. However as anticipated, this rises towards the end of the period analysed to more typically around ±20%. Some of the allocated oil uncertainties are considerably greater than these values (indeed some are considerably in excess of ±70% off the chart) at low Dumbarton flows.

Lochranza's uncertainty is determined from its measured dry oil uncertainty with some additional uncertainty due to the process shrinkage from MPFM to oil export conditions. As the water cut of P13 and P15 rises, Lochranza's allocated uncertainty steadily rises from around ±5% initially to in excess of ±20%.

Gas Allocation

For the gas allocation similar trends are observed. The produced gas flows[3] for Dumbarton and Locranza for the full 5½ year period analysed are presented in Figure 9:

Figure 9 –Dumbarton and Lochranza Allocated Produced Gas Mass and Lochranza Lift Gas



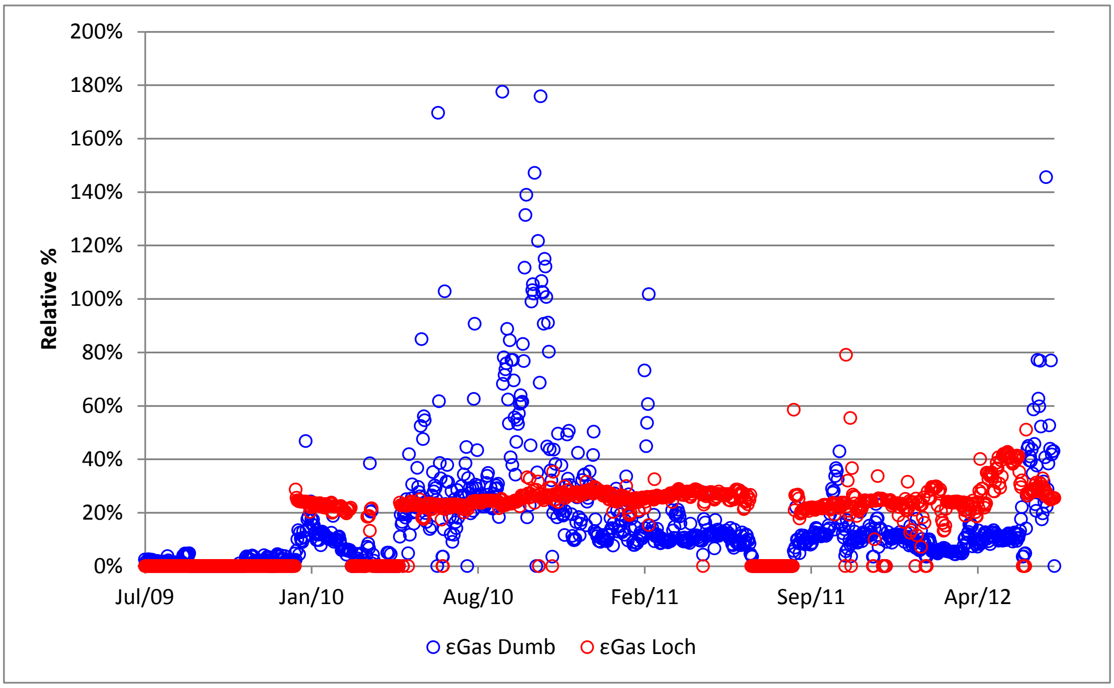
The gas production mimics the associated oil production since the GORs of both fields are stable.

Exacerbating the allocation uncertainty is the inclusion of lift gas in the Locranza MPFM measured gas rates which has to be netted off. (The uncertainty in each well's lift gas measurement increases the well's calculated produced gas uncertainty). In fact the lift gas starts to dominate the measured flow as indicated by the dashed red line in Figure 9.

The analogous gas allocation uncertainties are presented in Figure 10:

[3] Actual data has been filtered to smooth the data points for reasons of clarity.

Figure 10 – Lochranza and Dumbarton Allocated Gas Mass Relative Uncertainty



Again, the period presented includes 6 months of Dumbarton only when its gas allocation uncertainty was that of the combined gas product meters (between ±1% and ±5%).

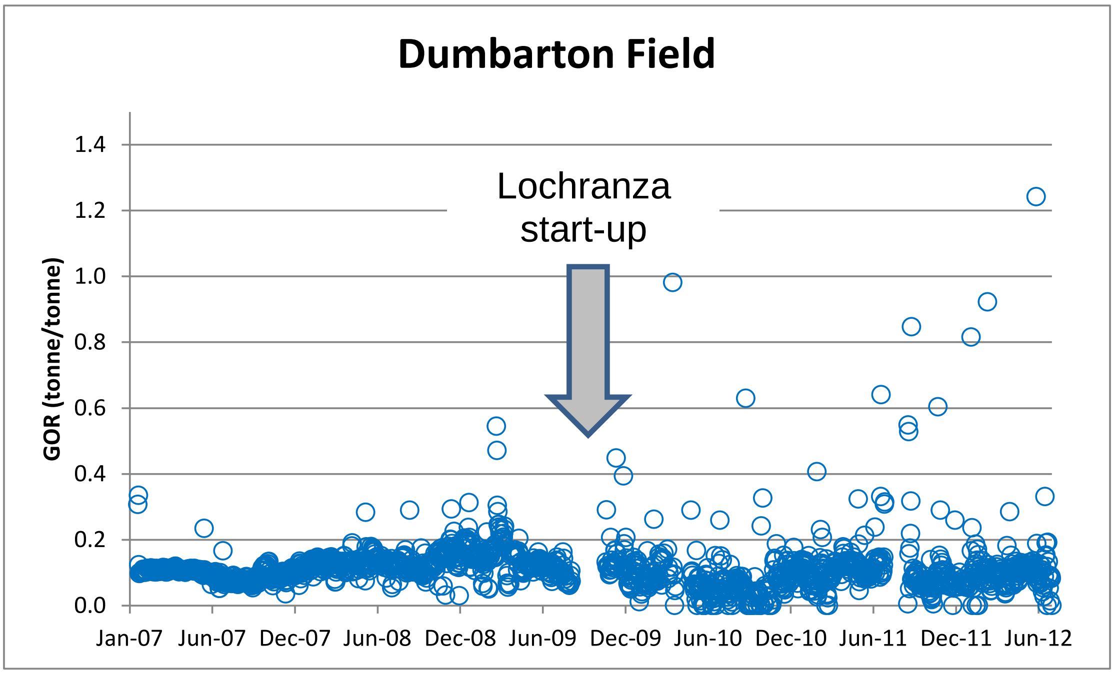
Similar to the oil allocation, once Lochranza starts up Dumbarton's allocated gas uncertainty experiences a step increase to between ±10% to ±20%, though the values can be considerably in excess of this, (in some cases exceeding 200% off the chart) at low Dumbarton flows.

Lochranza's allocated gas uncertainty is consistently above ±20% rising to around of ±40% at the end of the period when the lift gas routed through the MPFMs is dominating the gas flow.

Allocated GOR

As stated above in Section 2.3, the allocated GORs should provide a metric with which to monitor the performance of the allocation system. Analysis of Dumbarton's allocated GOR shows an increase in variability after Lochranza starts up in January 2010, as illustrated in Figure 11:

Figure 11 – Dumbarton Allocated GOR (Mass Based)



Again some points after Lochranza start-up are off the chart in excess of a GOR of 1.5 tonne/tonne. Though perhaps not readily apparent, because of the large vertical axis scale, the spread of GORs below the average rises once Lochranza starts up. A more analytical approach to assess the variability change is to consider the statistcs. The statistics for the two periods pre- and post-Lochranza start-up are summarised in Table 4:

Table 4 – Allocated Dumbarton GOR Statistics

	Pre-Lochranza	Post-Lochranza
Average	0.121	0.104
Maximum value	0.545	2.666
Minimum value	0.030	0.000
Standard Deviation (abs)	0.044	0.153
Uncertainty (rel %)	±72%	±294%

Dumbarton's GOR has become more variable as indicated by the higher standard deviation. This is to be expected in accordance with the increased uncertainty in its allocated oil and produced gas.

Similar to Dumbarton, Lochranza should also exhibit a relatively stable GOR as it has a consistent wellstream composition (see Figure 5) and its reservoir is also above its bubble point.

Lochranza's allocated GOR is also presented in Figure 12 and the associated statistics summarised in Table 5.

Figure 12 – Lochranza Allocated GOR (Mass Based)

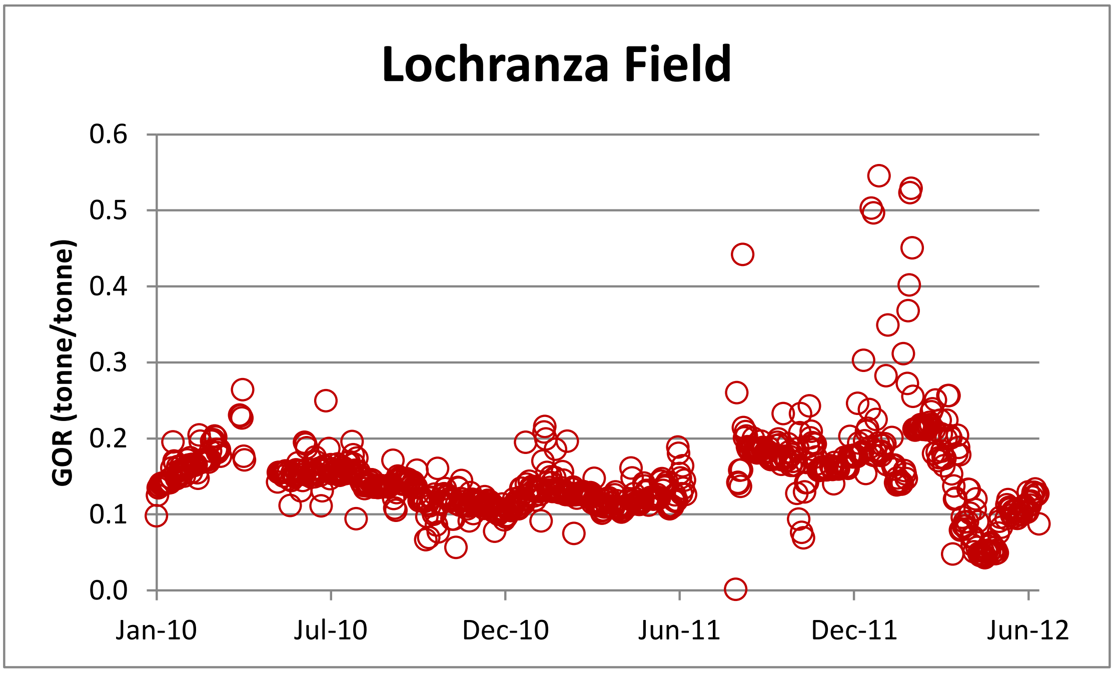


Table 5 – Allocated Lochranza GOR Statistics

	All Data	Outliers Removed
Average	0.155	0.144
Maximum value	2.965	0.303
Minimum value	0.001	0.001
Standard Deviation (abs)	0.138	0.040
Uncertainty (rel %)	±178%	±56%

Similar to Dumbarton, Lochranza does exhibit an ostensibly stable GOR with some variation probably due to process instability effects as was observed with Dumbarton.

The Grubbs' Test was applied to the Lochranza GORs and outliers removed from the data to calculate the average GOR of 0.144 tonne/tonne with an estimated uncertainty of ±56%.

2.5 An Uncertain Future

Concerns arose with the then incumbent Dumbarton By-Difference allocation scheme because the already increasing uncertainties in Dumbarton's allocated oil and gas and variability in allocated GOR were only anticipated to deteriorate further because:

- Dumbarton's production was declining
- Lochranza's MPFMs' dry oil measurement uncertainty was increasing with increasing water production
- Lochranza's MPFMs' gas measurement uncertainty was increasing with rising lift gas rates

- A new field, Balloch, was being tied back to GPIII.

The above factors meant that Dumbarton's fraction of the total production on GPIII would continue to fall and its already worsening allocation uncertainty would increase to unacceptable levels.

The rising uncertainty in Dumbarton's allocated oil and gas has consequences in that it provides poor data for reservoir modelling purposes. Indeed the increasing variability of the GOR was leading to credibility problems with the reservoir engineers, thus undermining the integrity of the allocation system.

An additional possibility, which occurred in a similar system, is the potential for a field to be shut-in due to its flare consent limit being breached. Due to a high gas allocation uncertainty (analogous to Dumbarton on GPIII), an over-allocation of produced gas was experienced by a low GOR field in this system. This resulted in an increase in its allocated flare gas which precipitated the very real threat of shut-in of production as it approached its flare consent limit.

The need to improve Dumbarton's allocation uncertainty resulted in alternative allocation schemes being considered and these are discussed in the next section.

3 ALTERNATIVE ALLOCATION APPROACHES

3.1 Pro Rata

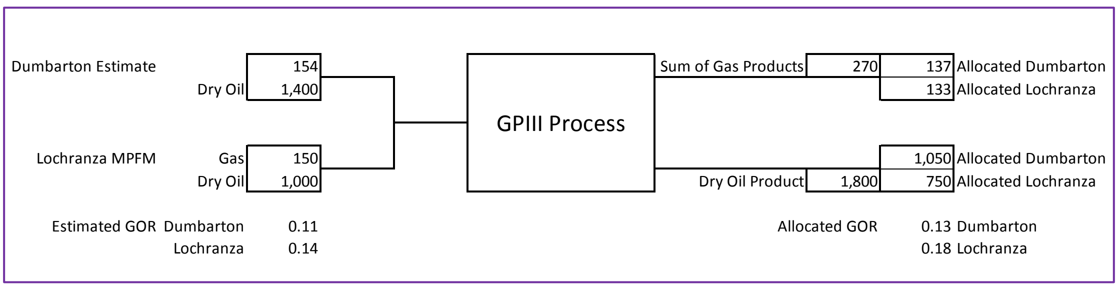
The most obvious alternative to allocating Dumbarton By-Difference is to allocate Pro Rata by incorporating Dumbarton's estimated production from well test information into the allocation scheme.

For example the oil can be allocated in proportion to:

- the sum of Lochranza's MPFM dry oil measured rates (after allowing for process shrinkage)
- the sum of Dumbarton wells' most recent tested dry oil rates (after allowing for hours in production and process shrinkage) – this is termed Dumbarton's estimated oil rate.

This is illustrated numerically again using the simple example:

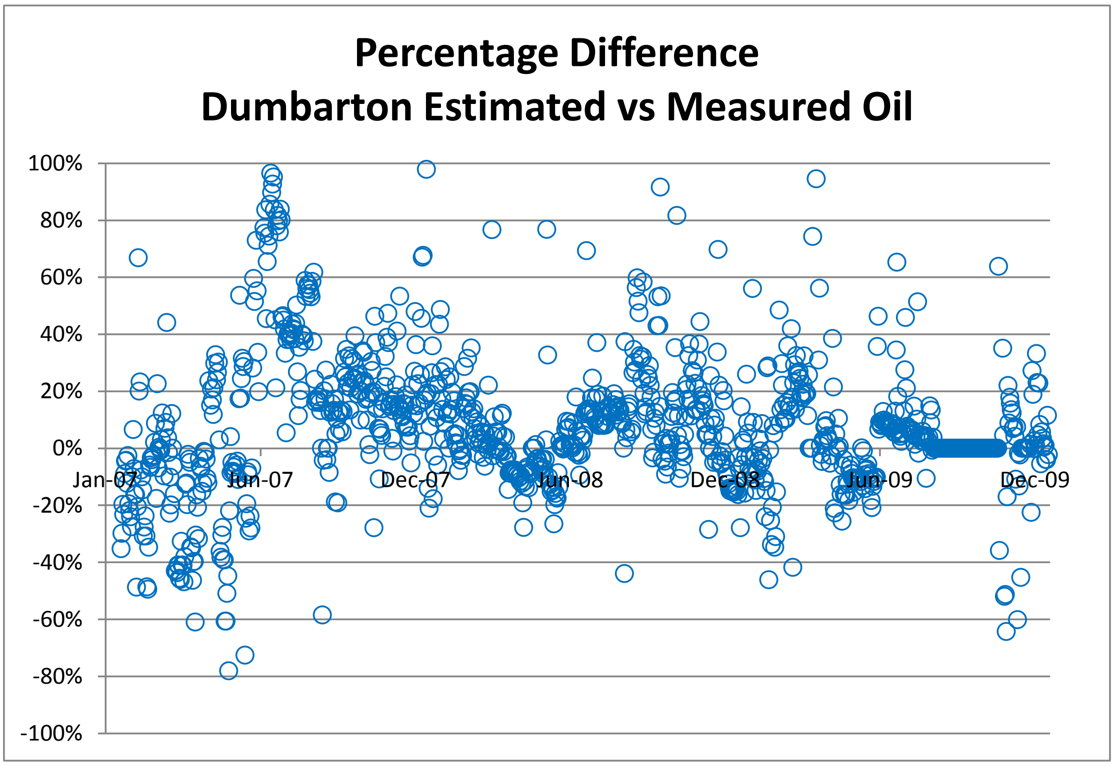
Figure 13 – Simplified Example: Pro Rata Allocation



The total oil product quantity is allocated in proportion to Lochranza's MPFM measured dry oil (1,000) and Dumbarton's estimated production based on well tests (1,400). Hence Lochranza is allocated 1,800 * 1,000 / (1,000 + 1,400) = 750 and Dumbarton 1,800 * 1,400 / (1,000 + 1,400) = 1,050. Gas is allocated on a similar basis.

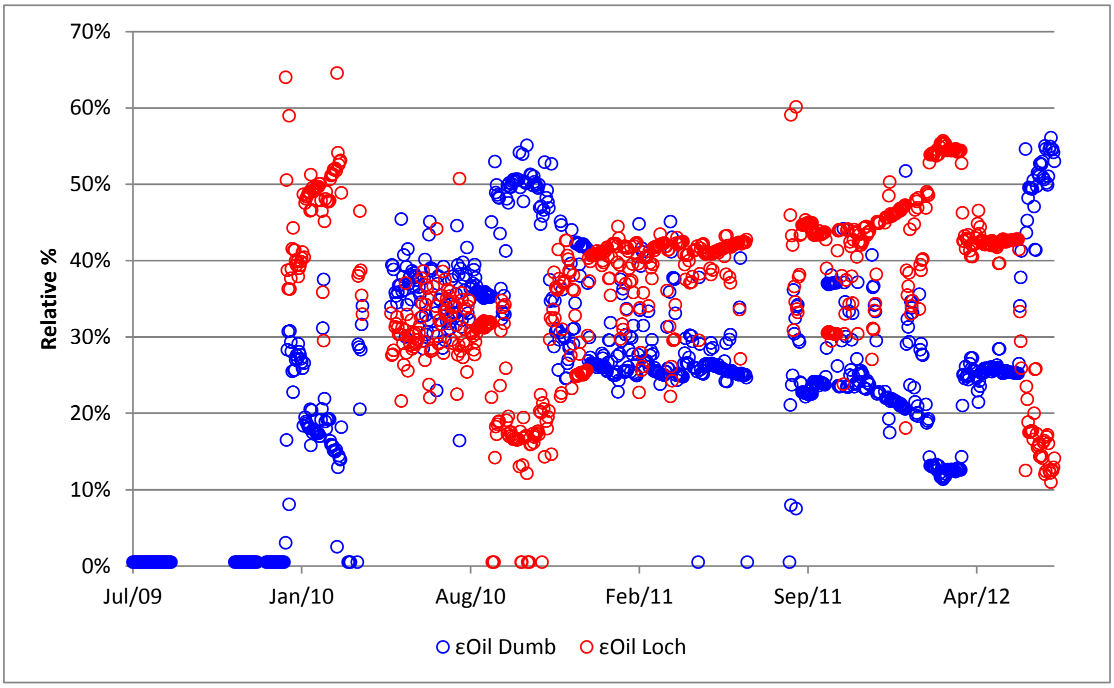
This approach has been applied to the real data and the analytical uncertainty in the allocated oil calculated. For this analysis, the uncertainty in the estimated Dumbarton oil production based on well tests is required. This can be deduced from a comparison of Dumbarton's estimated oil and the exported product oil for the period when Dumbarton only was on stream (Jan 2007 to Jan 2010). The percentage difference between Dumbarton's estimated oil rate and the actual measured oil production is plotted in Figure 14:

Figure 14 – Relative Difference between Dumbarton Estimated vs Measured Oil



The calculated uncertainty in Dumbarton's estimated oil is relatively poor at around ±67%. Using the analytical TSM approach the resultant uncertainty in the oil allocated to Dumbarton and Lochranza has been calculated and is presented in Figure 15:

Figure 15 – Lochranza and Dumbarton Pro Rata Allocated Oil Mass Relative Uncertainty



Comparison with the Dumbarton By-Difference allocation uncertainties presented in Figure 8 illustrates that the high uncertainty in the Dumbarton estimated oil significantly increases, not only the Lochranza allocated oil uncertainty, but also Dumbarton's.

Based on these oil uncertainty figures alone Pro Rata does not appear a viable option.

3.2 Using More Information

One of the key features of this system is the identification of the fact that the two fields' wellstream compositions are essentially constant resulting in a stable GOR. The GOR may vary from day to day depending on operating conditions and any process dynamical instabilities but it should not vary as widely as is observed in the allocated data. Indeed the variation in the allocated GOR has been used as a metric to judge the quality of, and consequently question, the allocation results.

The GORs connect the oil and gas allocated to both fields and because they can be estimated to within a tolerance or nominal uncertainty they can be incorporated as inputs into the allocation system. The method by which this has been implemented is described in the next section.

3.3 Uncertainty Based Allocation

Uncertainty based allocation has been previously described in several papers (for example [5], [6] and [7]) and actually applied in one North Sea oil allocation system [7]. Its superiority in terms of allocation uncertainty over By-Difference and Pro Rata approaches has also been discussed [8].

The additional effort and complexity associated with UBA is worthwhile in systems where measurements of the estimated production from two or more fields differ significantly in uncertainty rendering neither By-Difference nor Pro Rata allocation suitable over the full range of production. This is precisely the situation with the GPIII allocation. The additional feature of the inclusion of the fields' GORs as an input to the allocation system is also readily afford by UBA as it is based on data reconciliation techniques (described in [9]). Data reconciliation, as the name implies, involves the reconciliation of all relevant data to satisfy constraints such as mass balances, etc. It does this in a statistically optimal fashion by incorporating the uncertainty in the data.

The application of uncertainty based allocation to the GPIII system involves simultaneously reconciling estimated oil production, gas production and notional GOR associated with each field with the product oil and gas measured products. The inputs quantities to be reconciled are:

- Lochranza wells' MPFM dry oil (adjusted for processing effects)
- Sum of Dumbarton's wells' estimated oil production from well tests (adjusted for processing effects)
- Lochranza wells' MPFM gas (lift gas subtracted and adjusted for processing effects)
- Dumbarton's notional GOR
- Lochranza's notional GOR.

The UBA procedure takes these input quantities and adjusts them until there is a mass balance with measured product oil and gas. The adjusted or reconciled values are the allocated values. The adjustments are performed in such a way that the differences between the actual measured values and the reconciled (allocated) values is minimised (to be precise, the weighted sum of squares of the differences is minimised).

The technique takes into account the various uncertainties of each input. By incorporation of the input quantity accuracies the technique effectively gives more "weight" to those inputs which are expected to be more accurate. So for example, the approach still utilises the Dumbarton estimated oil based on well tests, which degraded the results of the Pro Rata allocation, but its influence on the allocation results is reduced because its uncertainty is relatively high and it is weighted accordingly. The Dumbarton estimated oil figure is worthy of inclusion, because at low Dumbarton flows its uncertainty may be better than that from the By-Difference approach.

It should be noted that the UBA approach described above is near statistically optimal. It would be statistically optimal if it also adjusted the product measurements. However, as any allocated quantities need to sum to the product measurements these have been excluded from any adjustments. This is appropriate because the product measurements have much lower uncertainties than the input quantities and therefore would experience little adjustment in the reconciliation procedure in any case.

The equations used to frame the allocation are:

Minimise (ψ) where:

$$\psi = \left(\frac{ADOILM_{P13} - THWOM_{P13}}{U_{THWOM_{P13}}}\right)^2 + \left(\frac{ADOILM_{P14} - THWOM_{P14}}{U_{THWOM_{P14}}}\right)^2$$

$$+ \left(\frac{ADOILM_{P15} - THWOM_{P15}}{U_{THWOM_{P15}}}\right)^2 + \left(\frac{ADOILM_{Dumb} - THWOM_{Dumb}}{U_{THWOM_{Dumb}}}\right)^2$$

$$+ \left(\frac{APGASM_{P13} - THWGM_{P13}}{U_{THWGM_{P13}}}\right)^2 + \left(\frac{AGOR_{P14} - THWGM_{P14}}{U_{U_{THWGM_{P14}}}}\right)^2$$

$$+ \left(\frac{APGASM_{P15} - THWGM_{P15}}{U_{THWGM_{P15}}}\right)^2 + \left(\frac{AGOR_{Loch} - NOTGOR_{Loch}}{U_{NOTGOR_{Loch}}}\right)^2$$

$$+ \left(\frac{AGOR_{Dumb} - NOTGOR_{Dumb}}{U_{NOTGOR_{Dumb}}}\right)^2 \quad (2)$$

Subject to the mass balance constraints on the oil phase and gas phase which are:

$$\Phi_{Oil} = 0 = -DOILM + ADOILM_{P13} + ADOILM_{P14} + ADOILM_{P15} + ADOILM_{Dumb} \quad (3)$$

$$\Phi 1_{Gas} = 0 = -TPGASM + APGASM_{P13} + APGASM_{P14} + APGASM_{P15} + \left(ADOILM_{Dumb} * AGOR_{Dumb}\right) \quad (4)$$

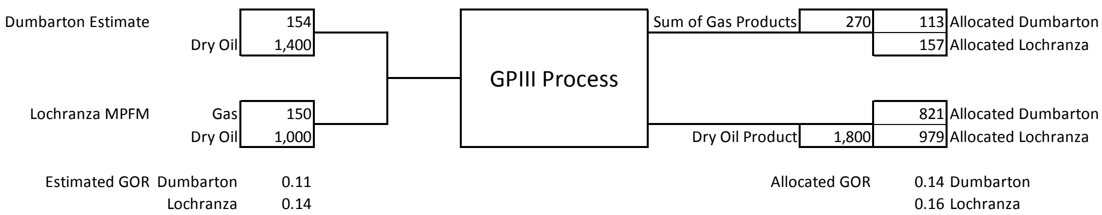
$$\Phi 2_{Gas} = 0 = APGASM_{P13} + APGASM_{P14} + APGASM_{P15} - \left(ADOILM_{Loch} * AGOR_{Loch}\right) \quad (5)$$

There are two gas constraints (equations (4) and (5)): the first ensures that the allocated Lochranza gas and Dumbarton allocated oil multiplied by GOR equals the total measured gas; the second ensures that the allocated Lochranza gas equals the Lochranza allocated oil multiplied by Lochranza GOR.

The fact that the two gas constraint equations include two of the quantities to be adjusted multiplied together (allocated oil and GOR) means that the problem is non-linear. The solution to these non-linear equations is more complex than for linear UBA (as described in [5], [6] and [7]) and an iterative technique is required.

For systems with two fields, the equations can be re-arranged as a series of simultaneous equations. However, for systems with more than two fields, the equations are more complex, and a matrix-solution method is recommended. The derivation and solution method used is described in Section 6. The UBA calculations have been carried out on Excel spreadsheets using recognised matrix algebra techniques. A specific matrix add-in for Excel was used to obtain the necessary capability and numerical precision [10].

Returning to the simple example to illustrate the non-linear UBA calculations:



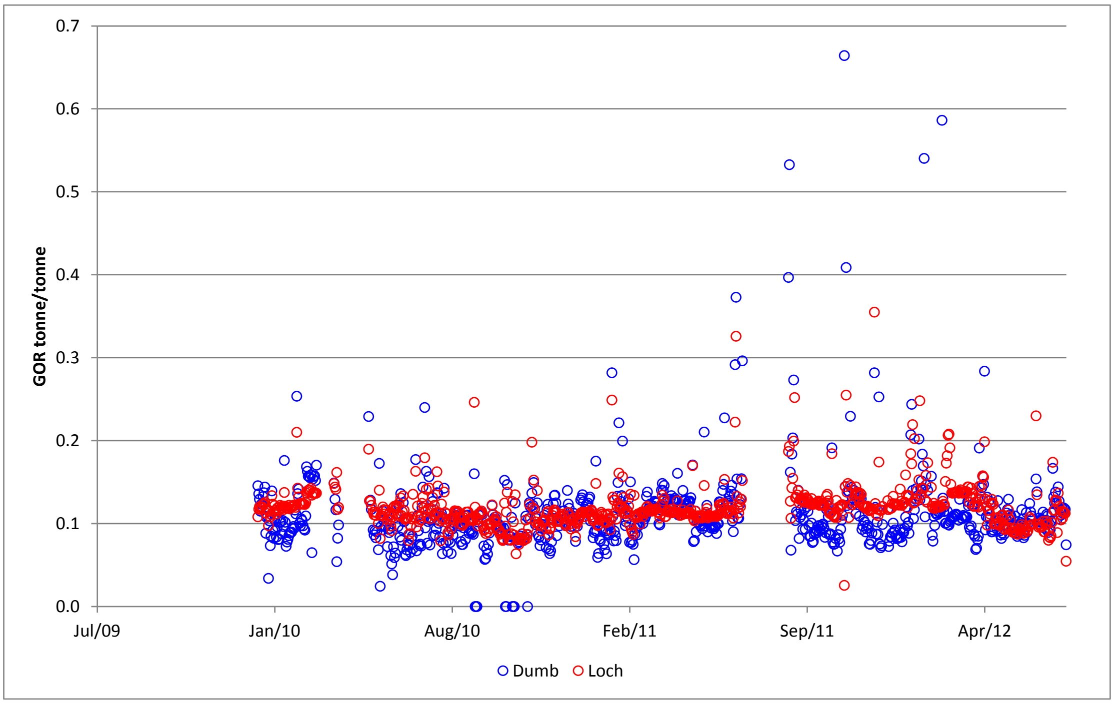
Inputs	**Measured**	**Uncertainty**	**Allocated**	**Difference**	**(Weighted Difference)^2**
Dumbarton Estimated Dry Oil	1,400	50%	821	-579	0.684
Lochranza MPFM Dry Oil	1,000	10%	979	-20.9	0.044
Lochranza MPFM Gas	150	20%	157	6.5	0.047
Dumbarton Estimated GOR	0.11	50%	0.14	0.028	0.264
Lochranza Estimated GOR	0.14	50%	0.16	0.020	0.080
					1.119 **Sum of squares**

The input variables have been adjusted to produce the reconciled, allocated values such that the square of sum of the weighted differences is minimised (red highlighted value of 1.119), whilst ensuring the allocated oil sums to the measured dry oil product and likewise the allocated gas sums to the measured gas products and the allocated GORs are similarly consistent.

For example the Dumbarton estimated oil (1,400) has an uncertainty of ±50% (or ±700 absolute). Its allocated oil is 821, which is 579 less than its estimated value but when weighted this is equal to (1,400 – 821)/700 = 0.827, and squared = 0.684.

This approach has been applied to the real data and the allocated GOR is plotted in Figure 16:

Figure 16 – Lochranza and Dumbarton UBA Allocated GOR



This plot illustrates the reduction in the variability of the GORs, particularly for Dumbarton. (The GOR stability is considered to be a metric with which to assess the veracity of the allocation results). It also tends to allocate a slightly higher GOR to Lochranza compared with Dumbarton which is as expected.

The three allocation methods are compared more directly in Section 4.

4 COMPARISON OF METHODS

4.1 Summary of Allocated Quantities

Table 6 provides a summary of the total allocated quantities generated by all three methods for the period after Lochranza started up and both fields were flowing:

Table 6 – Allocated Totals for Period from 2nd Jan 2010 to 3rd July 2012

	By-Difference	Pro Rata		UBA	
Dumbarton Total Allocated Oil (tonnes)	1,471,382	1,337,269	-9.1%	1,467,962	-0.2%
Lochranza Total Allocated Oil (tonnes)	987,076	1,121,188	13.6%	990,496	0.3%
Dumbarton Total Allocated Gas (tonnes)	128,863			155,653	20.8%
Lochranza Total Allocated Gas (tonnes)	138,004			111,215	-19.4%
Dumbarton Average Allocated GOR (tonne/tonne)	0.104			0.113	
Lochranza Average Allocated GOR (tonne/tonne)	0.155			0.148	

The total allocated oil and gas quantities are presented in the first four rows. The percentage difference in these figures generated by the Pro Rata and UBA methods compared to the historical Dumbarton by-difference approach are also presented.

The allocated oil results show that the Pro Rata approach has a very significant impact on the results compared to the historical allocation. UBA however, results in a much smaller impact on allocated totals compared with the original. This is as expected since the Lochranza MPFM oil flow uncertainties are much better than those for Dumbarton estimated from well tests.

A more marked impact is observed in the allocated gas when comparing UBA with By-Difference. However, the average daily UBA allocated GOR is now more consistent with the PVT data. (NB. The GOR is calculated from the daily average and not simply total gas divided by total oil for the period). Pro Rata was not included in the gas allocation comparisons since the oil allocation was so poor and Pro Rata was therefore not considered further.

4.2 Allocated Oil Uncertainty

The allocation uncertainties for the three methods are compared for Dumbarton and Lochranza in Figure 17 and Figure 18 respectively.

Figure 17 – Dumbarton Oil Mass Allocation Uncertainties

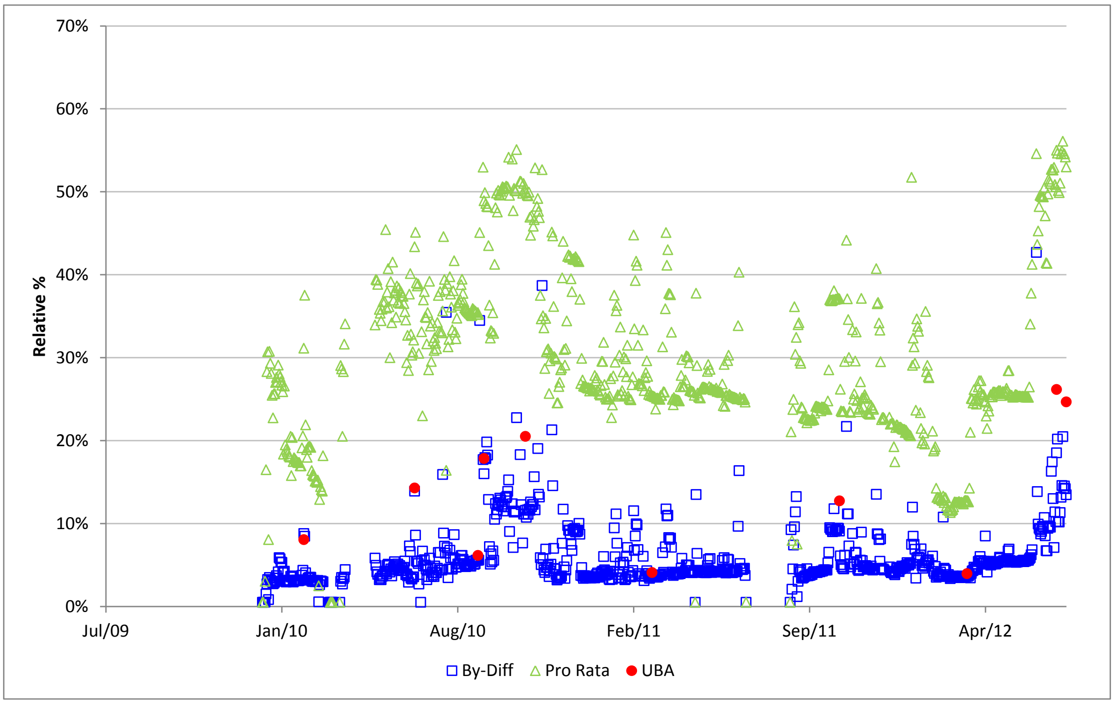
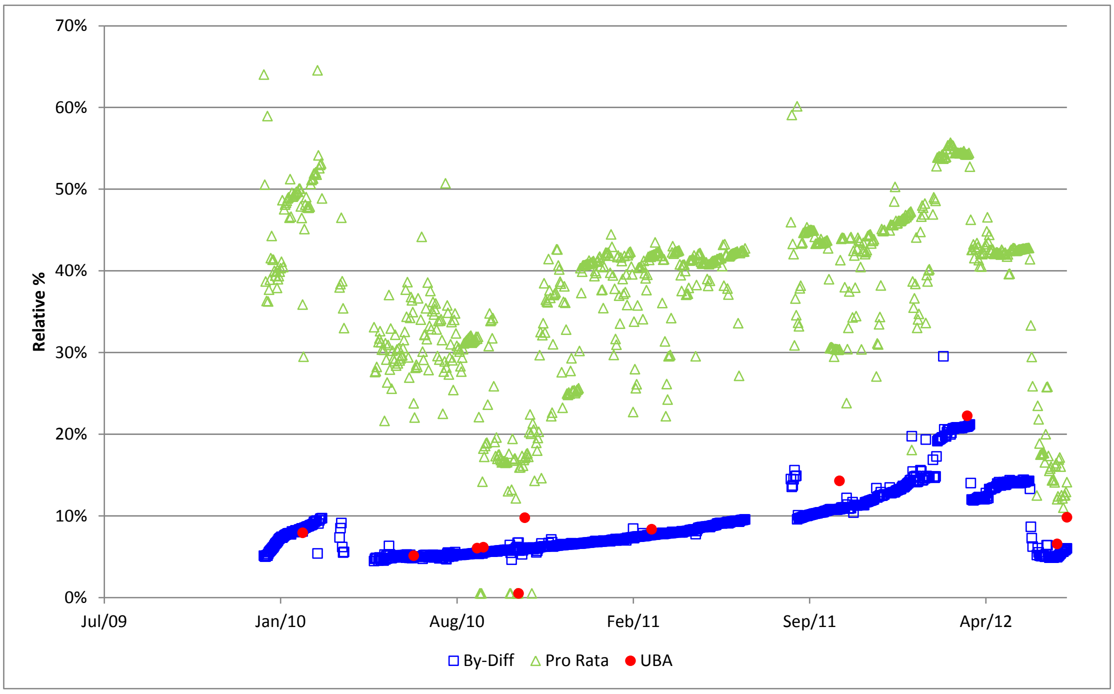


Figure 18 – Lochranza Oil Mass Allocation Uncertainties



There are only limited data points for the UBA method (indicated by the red dots) because the uncertainties had to be calculated using a relatively time consuming Monte Carlo method.

These plots illustrate similar uncertainties obtained using UBA compared with the historical By-Difference allocation method for both Dumbarton and Lochranza. Both methods' uncertainties are evidently much better than the poor allocation results generated by Pro Rata allocation.

4.3 Allocated Produced Gas Uncertainty

The gas allocation uncertainties for the By-Difference and UBA methods are compared for Dumbarton and Lochranza in Figure 19 and Figure 20 respectively.

Figure 19 – Dumbarton Produced Gas Mass Allocation Uncertainties

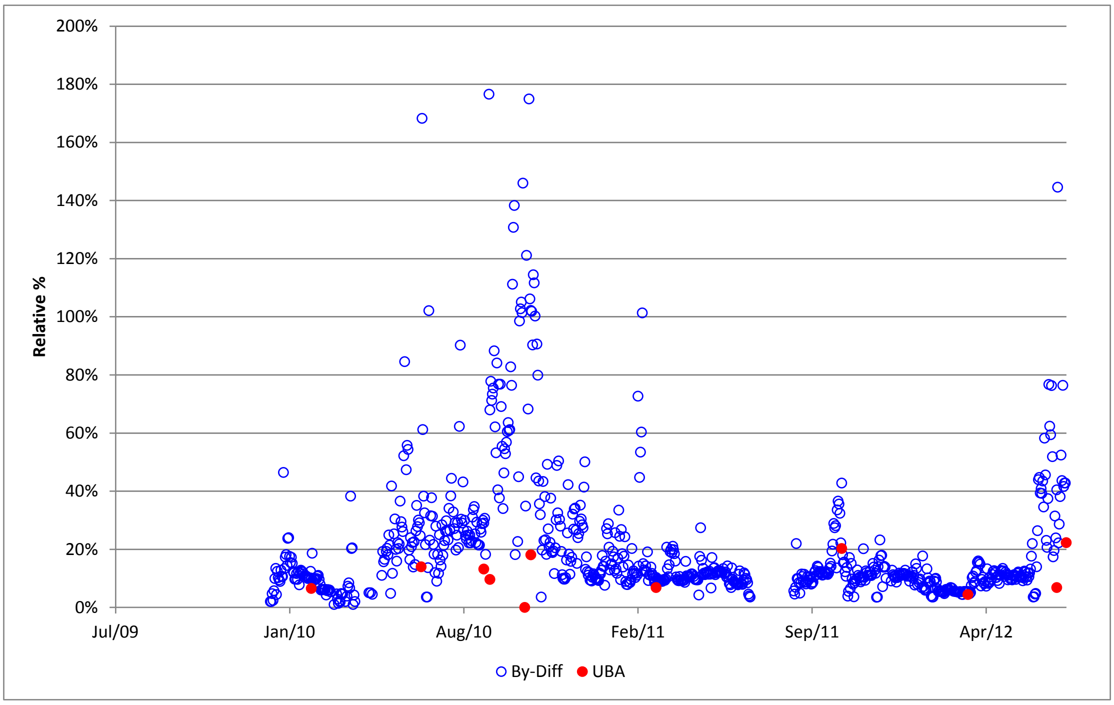
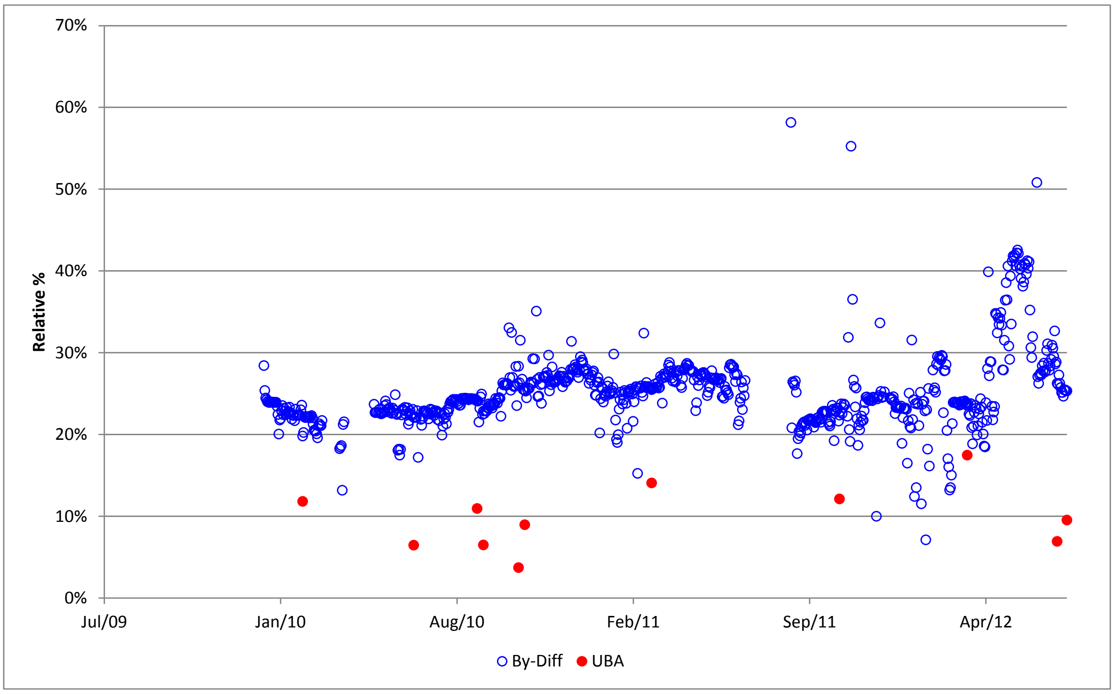


Figure 20 – Lochranza Produced Gas Mass Allocation Uncertainties



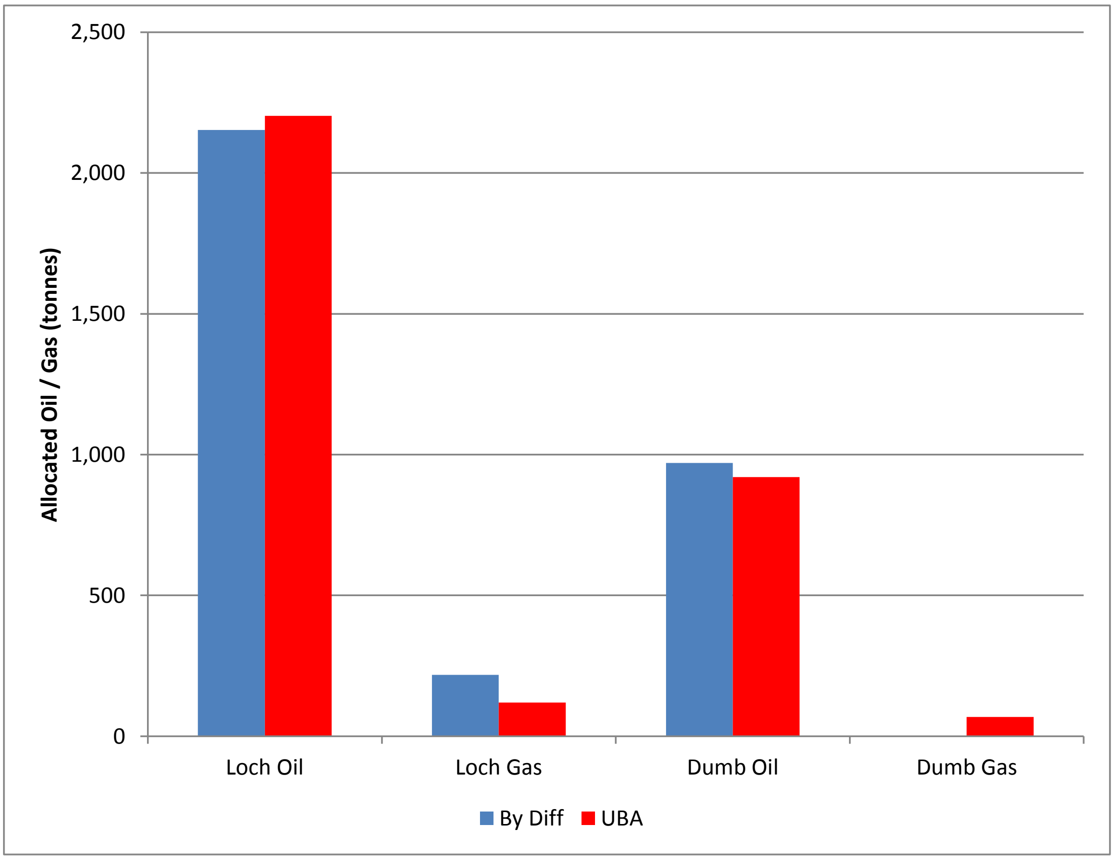
For the gas significant improvements in allocation uncertainty are observed with UBA compared to the historical By-Difference approach for both fields.

4.4 Anomalous Allocation Days

With the By-Difference allocation approach, there were a number of allocation days when Dumbarton was allocated some oil but zero gas, i.e. the measured Lochranza gas exceeds the total product gas. Application of UBA results in a more realistic allocation.

For example, this scenario occurred on the last day of the period analysed (2nd July 2012) and the allocation results using By Difference and UBA are summarised in Figure 21:

Figure 21 – Comparison of Allocation Results for By-Difference and UBA for 2nd July 2012



The blue series shows the By-Difference allocation results with Dumbarton being allocated zero gas even though it is allocated some oil, i.e. its allocated GOR is zero. The Dumbarton wells were flowing and hence it might be expected that Dumbarton would be allocated both gas and oil. By utilising the GOR data, the UBA method produces a more coherent, credible allocation (red series).

5 CONCLUSIONS

UBA offers a viable alternative allocation approach that:

- Utilises all pertinent data in the allocation, including the fields' ostensibly stable GORs and Dumbarton estimated well production (based on well tests);
- Allocates oil and gas simultaneously in a near statistically optimal fashion;
- Ensures Dumbarton's allocation uncertainty does not rise to unacceptable levels as its relative production declines, especially with new Lochranza wells and new fields starting up production;
- Produces oil allocation uncertainties comparable with the By-Difference and much better than Pro Rata approaches;
- Produces gas allocation uncertainties significantly better than the By-Difference approach;
- Results in more stable allocated GORs for both fields;

- Avoids anomalous allocation results, i.e. fields allocated oil but no gas;
- Produces more consistent, coherent and credible allocation results.

6 MATHEMATICAL ANALYSIS UBA MATRIX SOLUTION TECHNIQUE

This method is described in [11] and [12] and is based upon the principles of data reconciliation as described more generally in [9].

It should be noted that a rigorous data reconciliation method would reconcile the product measurements (DOILM, EXPGM, etc.) as well as the allocated quantities ($ADOILM_g$, etc). The sum of allocated quantities would not, therefore, exactly equal the recorded measurements. For allocation, it is generally required that the sum of the allocated quantities is equal to the recorded measurement. So although the product (fiscal) measurements are included in the equations below their uncertainties are assumed to tend to zero, and this ensures the sum of the allocated quantities is equal to the recorded measurement. This is justifiable because the fiscal product measurements are generally substantially more accurate than production estimates.

It should also be noted that total produced gas, with its associated uncertainty, represents the combined fiscal gas export, fuel, flare and injection gas streams less import gas, and their uncertainties. The total produced gas term can be replaced by the individual stream quantities and their associated uncertainties in the following equations. This simply leads to matrices of higher dimension in the equations. The entries representing fiscal export, fuel and flare gas would all be analogous to those for total produced gas shown here. Similarly, water could be included in the data reconciliation, with an additional constraint and inclusion of the necessary metered or estimated stream masses and uncertainties leading to a further increase in the dimensions of the matrices involved in the equations.

Theory

The full system of equations to be solved for the GP III system is shown below:

$$\psi = \left(\frac{ADOILM_{P13} - THWOM_{P13}}{U_{THWOM_{P13}}}\right)^2 + \left(\frac{ADOILM_{P14} - THWOM_{P14}}{U_{THWOM_{P14}}}\right)^2$$

$$+ \left(\frac{ADOILM_{P15} - THWOM_{P15}}{U_{THWOM_{P15}}}\right)^2 + \left(\frac{ADOILM_{Dumb} - THWOM_{Dumb}}{U_{THWOM_{Dumb}}}\right)^2$$

$$+ \left(\frac{APGASM_{P13} - THWGM_{P13}}{U_{THWGM_{P13}}}\right)^2 + \left(\frac{AGOR_{P14} - THWGM_{P14}}{U_{U_{THWGM_{P14}}}}\right)^2$$

$$+ \left(\frac{APGASM_{P15} - THWGM_{P15}}{U_{THWGM_{P15}}}\right)^2 + \left(\frac{AGOR_{Loch} - NOTGOR_{Loch}}{U_{NOTGOR_{Loch}}}\right)^2$$

$$+\left(\frac{AGOR_{Dumb} - NOTGOR_{Dumb}}{U_{NOTGOR_{Dumb}}}\right)^2$$

(6)

The mass balance constraints on the oil phase and gas phase are:

$$\Phi_{Oil} = 0 = -DOILM + ADOILM_{P13} + ADOILM_{P14} + ADOILM_{P15} + ADOILM_{Dumb}$$

(7)

$$\Phi 1_{Gas} = 0 = -TPGASM + APGASM_{P13} + APGASM_{P14} + APGASM_{P15} + \left(ADOILM_{Dumb} * AGOR_{Dumb}\right)$$

(8)

$$\Phi 2_{Gas} = 0 = APGASM_{P13} + APGASM_{P14} + APGASM_{P15} - \left(ADOILM_{Loch} * AGOR_{Loch}\right)$$

(9)

The optimum solution to the system is found by minimising the value of Ψ (psi) in Equation (6), subject to the constraints of Equations (7), (8) and (9).

For systems with two fields, simultaneous equations can be easily written out explicitly and solved iteratively. However, for systems with more than two fields, the equations are more complex, and a matrix-solution method is recommended. Such a solution is described below.

Matrix Solution Method – Inputs

The input data to the matrix solution method are provided in the form of arrays and vectors. The integer n represents the number of variables to be reconciled.

Y	(Input) vector of measured data (dimension n, 1).
X	(Calculated) vector of reconciled data (dimension n,1).
V	Variance-covariance matrix for Y (dimension n,n). The covariance of each element to itself is calculated from the square of the absolute uncertainty (U) of the measurement (Y_m) divided by 2, $(U_m/2)^2$. The covariance of any element with any other element is zero because the quantities are independent.
J	Jacobian matrix (dimension number of constraint equations n). This contains the coefficients of the derivatives of the oil and gas constraints Equations (7), (8) and (9) – see below for derivation.
P	Constraint Projection matrix (dimension n,n). This is used to enable the oil and gas mass balance constraints to be calculated using the Jacobian. It accounts for non-linear relationships while at the same time removing double-counting from the constraints. The matrix elements are 1 or 0 on the main diagonal according to which elements of the Jacobian and which measurements are to be used to

	derive the constraints. All off-diagonal elements are 0.

For example, for the 2-field Lochranza and Dumbarton application, the “measured” data comprised the stream measurements and the theoretical oil and gas production (at export conditions) for each Field. These were mass values, based on MPFM measurements for Lochranza Field and on well-tested oil quantities and constant GOR for Dumbarton Field. All theoretical production quantities were calculated within the allocation system.

For a higher-order system, such as GP III, the principles are the same but the matrices are extended to include the additional Field values.

The subsequent matrices are shown with only 2 Fields. Equivalent terms for additional Fields should be inserted where indicated by “…”.

$$Y=\begin{bmatrix} \text{DOILM} \\ \text{TPGASM} \\ \text{THWOM}_{\text{P13}} \\ \text{THWGM}_{\text{P13}} \\ \dots \\ \text{NOTGOR}_{\text{Loch}} \\ \text{THWOM}_{\text{Dumb}} \\ \text{NOTGOR}_{\text{Dumb}} \end{bmatrix} = \begin{bmatrix} \text{Metered Export Oil} \\ \text{Produced Gas} \\ \text{Theoretical P13 MPM oil (at export conditions)} \\ \text{Theoretical P13 MPM gas (at export conditions)} \\ \dots \\ \text{Notional GOR Lochranza (at export conditions)} \\ \text{Theoretical Dumbarton oil (at export conditions)} \\ \text{Notional GOR Dumbarton (at export conditions)} \end{bmatrix}$$

$$V=\begin{bmatrix} \left(\frac{U_{DOILM}}{2}\right)^2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \left(\frac{U_{TPRODGASM}}{2}\right)^2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \left(\frac{U_{THWOM,P13}}{2}\right)^2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{U_{THWGM,P13}}{2}\right)^2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{U_{NOTGOR,Loch}}{2}\right)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{U_{THWOM,Dumb}}{2}\right)^2 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \left(\frac{U_{NOTGOR,Dumb}}{2}\right)^2 \end{bmatrix}$$

$$P=\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix solution is an iterative method, based on the “Jacobian matrix” (J). The Jacobian terms reflect the non-linear terms in the least-squares-type method used to determine the minimum value of Ψ in Equation (6). The Jacobian terms represent the coefficients of the derivatives of the oil and gas constraints (Equations (7), (8) and (9)) with respect to each reconciled quantity, X_m , e.g., $\partial\Phi_{Oil}/\partial X_m$ and . $\partial\Phi 1_{Gas}/\partial X_m$.

$$J = \begin{bmatrix} \frac{\partial\Phi Oil}{\partial DOILM} & \frac{\partial\Phi Oil}{\partial TPRODGASM} & \frac{\partial\Phi Oil}{\partial ADOILM_{P13}} & \frac{\partial\Phi Oil}{\partial APGASM_{P13}} & \cdots & \cdots & \cdots & \cdots & \frac{\partial\Phi Oil}{\partial AGOR_{Loch}} \\ \frac{\partial\Phi 1Gas}{\partial DOILM} & \frac{\partial\Phi 1Gas}{\partial TPRODGASM} & \frac{\partial\Phi 1Gas}{\partial ADOILM_{P13}} & \frac{\partial\Phi 1Gas}{\partial APGASM_{P13}} & \cdots & \cdots & \cdots & \cdots & \frac{\partial\Phi 1Gas}{\partial AGOR_{Loch}} \\ \frac{\partial\Phi 2Gas}{\partial DOILM} & \frac{\partial\Phi 2Gas}{\partial TPRODGASM} & \frac{\partial\Phi 2Gas}{\partial ADOILM_{P13}} & \frac{\partial\Phi 2Gas}{\partial APGASM_{P13}} & \cdots & \cdots & \cdots & \cdots & \frac{\partial\Phi 2Gas}{\partial AGOR_{Loch}} \end{bmatrix}$$

$$J = \begin{bmatrix} -1 & 0 & 1 & 0 & \dots & \dots & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & \dots & \dots & 0 & AGOR_{Dumb} & ADOILM_{Dumb} \\ 0 & 0 & AGOR_{Loch} & -1 & \dots & \dots & APGASM_{Loch} & 0 & 0 \end{bmatrix}$$

The matrix solution is therefore an iterative method, because some of the coefficients of the non-linear Jacobian terms ($AGOR_{Dumb}$, $ADOILM_{Dumb}$, $AGOR_{Loch}$ and $ADOILM_{Loch}$) are dependent on the previous solution.

For the first iteration only, the Jacobian matrix uses the theoretical estimates of the non-metered Field GOR and Oil.

Matrix Solution Method

The reconciled measurements X which result in the minimum value of Ψ in the system of equations described above may be described as follows and are calculated using the method described in [11] and shown in Equation (10) below:

$$X = \begin{bmatrix} ADOILM \\ ATPRODGASM \\ ADOILM_{P13} \\ APGASM_{P13} \\ \dots \\ \dots \\ \dots \\ \dots \\ AGOR_{Dumb} \end{bmatrix}$$

$$X = Y - K\left(f\left(X_0\right) + J\left(Y - X_0\right)\right) \quad (10)$$

Where,

X is the vector containing the reconciled measurements calculated by this iteration.
Y is the vector containing the initial measurements, as defined above.
K is an intermediate matrix, defined as:

$$K = VJ^T\left(JVJ^T\right)^{-1} \quad (11)$$

V is the covariance matrix for **Y**, as defined above.
$\mathbf{J}^T$ is the transpose of the Jacobian matrix, **J.**
$\mathbf{f(X_0)}$ is the imbalance vector, and is calculated from the product of the Jacobian matrix (**J**), the Constraint projection matrix (**P**) and the current estimated measurements ($\mathbf{X_0}$).

$$f(X_0) = JPX \tag{12}$$

J is the Jacobian matrix, **P** is the Constraint Projection matrix, as defined above.
$\mathbf{X_0}$ is the vector containing the reconciled measurements from the previous iteration.

Matrix Solution Method – Initialisation

1. Specify elements of measurements matrix, **Y**.
2. Calculate elements of variance-covariance matrix, **V**.
3. Specify initial elements of initial Jacobian matrix, $\mathbf{J_1}$, using the theoretical field quantities.
4. Calculate intermediate matrix K from Equation (11): $\mathbf{K=V\ J^T\ (J\ V\ J^T)^{-1}}$.
5. Initialise value of reconciled measurements vector, $\mathbf{X_0 = Y}$.
6. Calculate new values of reconciled measurements vector, **X** from Equations (10) and (12).

Matrix Solution Method – Iteration

7. Update value of reconciled measurements vector, $\mathbf{X_0 = X}$ from previous iteration.
8. Update elements of Jacobian matrix, **J**, using the latest reconciled measurements.
9. Update intermediate matrix K from Equation (11): $\mathbf{K=V\ J^T\ (J\ VJ^T)^{-1}}$.
10. Calculate new values of reconciled measurements vector, **X** from Equations 4 and 6.
11. Calculate absolute change in reconciled measurements vector: $\mathbf{ABS(X- X_0)}$.
12. If the sum of the absolute changes in reconciled measurements has changed by more than the specified tolerance, repeat steps 7 to 12.

NOTATION

ADOILM	Allocated dry oil mass
AGOR	Allocated GOR
APGASM	Allocated produced gas mass
DOILM	Measured dry product oil mass
e	Uncertainty (absolute)
EXPGM	Export gas mass
$f(X_0)$	Imbalance vector
J	Jacobian matrix
K	Intermediate matrix
NOTGOR	Notional GOR
P	Constraint projection matrix
THWGM	Theoretical well gas mass
THWOM	Theoretical well oil mass
TPGASM	Total produced gas mass
U	Absolute uncertainty
V	Variance covariance matrix
WLR	Water Liquid Ratio
X	Reconciled or allocated data vector
X_0	Reconciled or allocated data from previous iteration vector
Y	Input data vector

Greek

ε	Uncertainty (relative)
ψ	Objective function
ϕ	Constraint

Subscripts

Dumb	Dumbarton
liq	mass of liquid
Loch	Lochranza
moil	mass of oil
P13, etc	Well P13, etc

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